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Least *p*th Powers of Deviations

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1. INTRODUCTION

Let R denote the set of real numbers. If $x_1, x_2, \ldots, x_n$ is a finite sequence of points in R, then, as x ranges over R, $\sum_{k=1}^{n} (x_k - x)^2$ is minimal if and only if x is equal to the arithmetic mean of the numbers $x_1, x_2, \ldots, x_n$. This simple observation is the point of departure in Gauss's important "method of least squares." Gauss also suggested using other powers of the deviations [11, pp. 5, 135].

Let X be a real Hilbert space or a finite-dimensional, real, rotund normed linear space (see below); let $x_1, x_2, \ldots, x_n$ be a finite sequence of points in X. Let $1 \leqslant p < \infty$. For every $x \in X$, set $S_p(x) = \sum_{k=1}^{n} \| x_k - x \|^p$ and $A_p(x) = \{(1/n) \sum_{k=1}^{n} \| x_k - x \|^p\}^{1/p}$. Also, let $l(p) = \inf\{S_p(x): x \in X\}$ and $m(p) = \inf\{A_p(x): x \in X\}$. Finally, let $m(\infty) = \inf\{A_\infty(x): x \in X\}$, where, for every $x \in X$, $A_\infty(x) = \max\{\| x_k - x \|: 1 \leqslant k \leqslant n\}$. If $1 < p \leqslant \infty$, then, as we prove below, the infimum $m(p)$ is attained at a unique point $x(p) \in X$.

In this paper, the least pth powers of deviations are investigated; that is, $l(p)$ is studied. For certain technical reasons, it is convenient to consider an equivalent problem, namely, that of minimizing the pth order average, $A_p(x)$, of the distances $\| x_1 - x \|, \| x_2 - x \|, \ldots, \| x_n - x \|$ from x to each of the points x_k. An additional advantage is that A_p admits a generalization in which the counting measure on $\{x_1, x_2, \ldots, x_n\}$ is replaced by a finite (nonnegative) Borel measure on a compact subset of X. We shall study various qualitative and quantitative aspects of $l(p)$, $m(p)$, and $x(p)$, including their behavior as $p \to 1+$ and as $p \to \infty$. For example, we prove that $m(p) \nearrow m(\infty)$ as $p \to \infty$. Moreover, convexity properties of S_p and A_p are determined.

If $X = R$ and n is odd, then, in the phraseology of statistics, $x(1)$ is the median of the sequence $x_1, x_2, ..., x_n$ [17, p. 85; 2, p. 32], $m(1)$ is the mean deviation from the median, $x(2)$ is the arithmetic mean [2, p. 36], and $m(2)$ is the standard deviation; further, $m(\infty)$ is associated with Laplace's method of minimal approximation, which he devised in 1799 [24, p. 259]. For a general value of p, $x(p)$ and $m(p)$ are the simultaneous maximum likelihood estimates of the location and scale parameters, respectively, based on an independent sample $x_1, x_2, ..., x_n$ taken from a parent population known to have a "modified normal distribution" in the sense of Subbotin [16, pp. 33–34]. Gentleman [12] studied the robust estimation of multivariate location by minimizing the sum of the pth powers of the deviations. Among other things, he devised an efficient algorithm for computing the estimator. Since he dealt with Euclidean distance raised to the pth power, his work is an elaboration of a special case of Huber's class of estimators. For a general X and a general $p(1 < p \leq \infty)$, the point $x(p)$ locates a central position relative to the points $x_1, x_2, ..., x_n$, and $m(p)$ measures the dispersion (variation, scattering) of the points.

If X is Euclidean 3-space R^3, and if $x_1, x_2, ..., x_n$ are distinct points of a plane in R^3, then, for each x in the plane, $S_2(x)$ is the moment of inertia about the axis in X perpendicular to the plane at x of the system consisting of unit masses at the points x_k (each x_k endowed with mass 1). By the discrete case of Steiner's transfer theorem of mechanics [7, p. 439], $x(2) = (1/n) \sum_{k=1}^{n} x_k$. Also, $A_2(x)$ is the radius of gyration of the system about that axis, and $m(2)$ is $A_2(x)$ for x, the center of mass of the system.

The case $p = 1$ exhibits certain irregularities that are not present when $1 < p < \infty$. For example, if $x_1, x_2, ..., x_n$ are real numbers, then $S_1(x)$ is minimum whenever x is a median of the x_k, but a median is generally not unique if n is even [2, pp. 32–34]. For this reason, we give the case $p = 1$ a special treatment. When $X = R^2$, $p = 1$, and $n = 3$, the minimization of $S_1(x)$ is a problem in geometric inequalities posed by Fermat [10, pp. 21–23] and solved (for arbitrary n) by Steiner [9, pp. 354–360]. (Melzak [19, p. 140] suggests that Cavalieri was the first to pose and solve the problem for $n = 3$.) For $n = 3$, the problem can be solved in a simple way both mechanically (by a contrivance using strings and weights [23, pp. 113–117]) and geometrically [19]; a limiting case of the modified isoperimetric problem also yields the result [9, p. 379].

In the general case, it turns out that the behavior of $l(p)$ for large values of p depends directly on $m(\infty)$. If $X = R$, then $x(\infty)$ is simply the midpoint of the convex hull of $\{x_1, x_2, ..., x_n\}$ and we can determine the limiting behavior of $l(p)$ completely. The limiting behavior of $x(p)$ in the case $X = R$ was determined by Jackson [15] in 1921.

We recall that a normed linear space is strictly normalized if $x \neq 0$,

$y \neq 0$, and $\| x + y \| = \| x \| + \| y \|$ imply that $y = \alpha x$ for some $\alpha > 0$ [1, pp. 11–12]. Finite-dimensional Euclidean spaces, inner-product spaces [4, p. 32; 25, p. 122], and the Lebesgue spaces $L_p(Y, \mathscr{A}, \mu)$, where $(Y, \mathscr{A}, \mu)$ is an arbitrary measure space and $1 < p < \infty$, are all strictly normalized [14, p. 192]; but $L_1(0, 1)$ is not. In particular, the finite-dimensional normed linear space l_p^n, consisting of all n-tuples $x = (\alpha_1, \alpha_2, ..., \alpha_n)$ of real numbers with the norm $\| x \|_p = \{\sum_{k=1}^n | \alpha_k |^p\}^{1/p}$, is strictly normalized if $1 < p < \infty$ and $n = 1, 2, ...$. However, neither l_1^n nor l_∞^n, where

$$\| x \|_\infty = \max\{| \alpha_1 |, | \alpha_2 |, ..., | \alpha_n |\},$$

is strictly normalized if $n = 2, 3, ...$ For l_1^n, consider $x = (1, 0, 0, ..., 0)$ and $y = (0, 1, 1, ..., 1$; as to l_∞^n, consider $x = (1, 1, 0, ..., 0)$ and $y = (-1, 1, 0, ..., 0)$.) A normed linear space is strictly normalized if and only if its closed unit ball is strictly convex; in other words, a strictly normalized space is a rotund, or strictly convex, space [18, pp. 138–139; 25, p. 111]. A finite-dimensional normed linear space is rotund if and only if it is uniformly convex [25, pp. 109, 111].

2. The Main Theorems

We are now ready to prove some theorems about $S_p(x)$, $A_p(x)$, $x(p)$, $l(p)$, and $m(p)$.

Lemma 1. *Let $x_1, x_2, ..., x_n$ be a finite sequence of points in a real Hilbert space X; let H be the convex hull of $\{x_1, x_2, ..., x_n\}$; and let $x \in X - H$. Then H is compact* [5, p. 138], *there exists a unique point $x^* \in H$ such that $\| x - x^* \| = \inf\{\| x - y \| : y \in H\}$* [4, p. 68], *and $\| y - x^* \| < \| y - x \|$ for each $y \in H$.*

Proof. It is known [22] that if a point z of a real Euclidean space E does not belong to the convex hull S^* of a nonempty compact subset S of E, then the point z^* of S^* closest in S^* to z is closer than z to every point of S.

Since H is contained in the Euclidean space spanned by $x, x_1, x_2, ..., x_n$, the desired conclusion follows from the result.

Theorem 1. *Let X be a real Hilbert space or a finite-dimensional, real, rotund normed linear space; let $x_1, x_2, ..., x_n$ be a finite sequence of points in X; and let $1 \leqslant p < \infty$. Then there exists a point $x(p) \in X$ such that $S_p(x(p)) = l(p)$, that is, such that $A_p(x(p)) = m(p)$. Moreover, if X is a Hilbert space or is two-dimensional, then each such $x(p)$ is in the convex hull of $\{x_1, x_2, ..., x_n\}$. If $1 < p < \infty$, then $x(p)$ is unique.*

Proof. We exclude (as we may) the trivial case $x_1 = x_2 = \cdots = x_n$. Once again, let H denote the convex hull of $\{x_1, x_2, ..., x_n\}$. Also, assume $1 < p < \infty$.

First, assume that X is a Hilbert space.

If $x \in X - H$, then let x^* denote the unique point of H that is closest to x. According to Lemma 1, $\| y - x^* \| < \| y - x \|$ for each $y \in H$. Hence, $\sum_{k=1}^n \| x_k - x^* \|^p < \sum_{k=1}^n \| x_k - x \|^p$. This proves that it suffices to minimize $A_p(x)$ as x ranges over H. Since A_p is continuous on the compact set H, the infimum $m(p)$, of $A_p(x)$ as x ranges over X, is attained at a point $x(p) \in H$.

To prove that $x(p)$ is unique, suppose that $x', x'' \in X$ and that $m(p) = A_p(x') = A_p(x'')$. Then, by Minkowski's inequality,

$$\begin{aligned}
&A_p((1/2)(x' + x''))\\
&\quad = \left\{(1/n) \sum_{k=1}^n \|(1/2)(x_k - x') + (1/2)(x_k - x'')\|^p\right\}^{1/p}\\
&\quad = (1/2)(1/n)^{1/p} \left\{\sum_{k=1}^n \|(x_k - x') + (x_k - x'')\|^p\right\}^{1/p}\\
&\quad \leqslant (1/2)(1/n)^{1/p} \left\{\sum_{k=1}^n (\| x_k - x' \| + \| x_k - x'' \|)^p\right\}^{1/p}\\
&\quad \leqslant (1/2)(1/n)^{1/p} \left[\left\{\sum_{k=1}^n \| x_k - x' \|^p\right\}^{1/p} + \left\{\sum_{k=1}^n \| x_k - x'' \|^p\right\}^{1/p}\right]\\
&\quad = (1/2)\{A_p(x') + A_p(x'')\}\\
&\quad = m(p).
\end{aligned}$$

Now, $m(p) \leqslant A_p((1/2)(x' + x''))$ by the definition of $m(p)$; hence, equality signs hold in the last three inequalities. Therefore, since X is strictly normalized, there exist, for $k = 1, 2, ..., n$, nonnegative real numbers c_k and d_k such that $c_k + d_k > 0$ and $c_k(x_k - x') = d_k(x_k - x'')$. Since equality occurs in Minkowski's inequality, there exist nonnegative real numbers c and d such that $c + d > 0$ and $c \| x_k - x' \| = d \| x_k - x'' \|$ for $k = 1, 2, ..., n$. From this and $m(p) = A_p(x') = A_p(x'') > 0$, we conclude that $c = d > 0$ and for each k, $\| x_k - x' \| = \| x_k - x'' \|$; thus, if $x_k - x' \neq 0$, then $c_k \| x_k - x' \| = d_k \| x_k - x'' \| = d_k \| x_k - x' \|$, $c_k = d_k > 0$, $x_k - x' = x_k - x''$, and $x' = x''$.

Next, suppose that X is a finite-dimensional, real, rotund normed linear space. Let

$$\| x_m \| = \max\{\| x_k \|: 1 \leqslant k \leqslant n\},$$

and let $K = \{x \in X: \| x \| \leqslant 2 \| x_m \|\}$. Since $A_p(0)$ and $A_p(x)$ are averages of distances, it is geometrically obvious that $A_p(0) < A_p(x)$ if $x \in X - K$. To prove this, note that if $x \in X - K$, then for each k, $\| x_k \| \leqslant \| x_m \| = 2 \|x_m \| - \| x_m \| < \| x \| - \| x_k \| \leqslant \| x_k - x \|$. Hence,

$$\left\{(1/n) \sum_{k=1}^{n} \| x_k \|^p\right\}^{1/p} < \left\{(1/n) \sum_{k=1}^{n} \| x_k - x \|^p\right\}^{1/p},$$

that is, $A_p(0) < A_p(x)$ if $x \in X - K$.

Since K is a closed bounded subset of the finite-dimensional normed linear space X, K is compact. As A_p is continuous on the compact set K, there exists a point $x(p) \in K$ such that $A_p(x(p)) \leqslant A_p(x)$ whenever $x \in K$. In particular, $A_p(x(p)) \leqslant A_p(0) < A_p(x)$ whenever $x \in X - K$. Hence, $A_p(x(p)) = \inf\{A_p(x): x \in X\}$. The proof that $x(p)$ is unique is the same as that for the previous case.

Finally, suppose that X is a two-dimensional, real, rotund normed linear space. We want to prove that $x(p) \in H$. Let $A \subseteq X$ and $u, v \in X$; then v is said to be pointwise closer than u to A provided $\| v - a \| < \| u - a \|$ for each $a \in A$. If no point of X is pointwise closer than u to A, then u is called a closest point to A. Phelps [20], proved that if A is a bounded subset of X, then the closure of the convex hull of A is the set of all closest points to A. Let $A = \{x_1, x_2, ..., x_n\}$. Since H is closed, H is the closure of the convex hull of A. Thus, H is equal to the set of all closest points to $\{x_1, x_2, ..., x_n\}$. Suppose that $x(p) \in X - H$. (The existence and uniqueness of $x(p)$ have already been established.) Then $x(p)$ is not a closest point to $\{x_1, x_2, ..., x_n\}$; consequently, there exists a point y, which need not be in H, that is pointwise closer than $x(p)$ to $\{x_1, x_2, ..., x_n\}$. Hence, $\| y - x_k \| < \| x(p) - x_k \|$ for $k = 1, 2, ..., n$; and

$$A_p(y) = \left\{(1/n) \sum_{k=1}^{n} \| x_k - y \|^p\right\}^{1/p} < \left\{(1/n) \sum_{k=1}^{n} \| x_k - x(p)\|^p\right\}^{1/p}$$
$$= A_p(x(p)) = \inf\{A_p(x): x \in X\},$$

a contradiction. The case $p = 1$ is left to the reader.

THEOREM 2. *Let X be a real Hilbert space or a finite-dimensional, real, rotund normed linear space; and let $x_1, x_2, ..., x_n$ be a finite sequence of points in X. Then there exists a unique point $x(\infty) \in X$ such that $A_\infty(x(\infty)) = m(\infty)$. Moreover, if X is a Hilbert space or is two-dimensional, then $x(\infty)$ is in the convex hull of $\{x_1, x_2, ..., x_n\}$.*

Proof. We exclude (as we may) the trivial case $x_1 = x_2 = \cdots = x_n$.

Except for uniqueness, our conclusions can be proved in the same way as the corresponding conclusions of Theorem 1.

To prove that $x(\infty)$ is unique, suppose that $x', x'' \in X$ and that $A_\infty(x') = A_\infty(x'') = m(\infty)$. Then there exists an integer j such that $A_\infty((1/2)(x' + x'')) = \|(1/2)(x_j - x') + (1/2)(x_j - x'')\| \leqslant (1/2)\| x_j - x' \| + (1/2)\| x_j - x'' \| \leqslant (1/2)\, m(\infty) + (1/2)\, m(\infty) = m(\infty)$. From the definition of $m(\infty)$, we also have $m(\infty) \leqslant A_\infty((1/2)(x' + x''))$. Hence, equality signs hold in the last three inequalities. Consequently, $\| x_j - x' \| = m(\infty) = \| x_j - x'' \|$. If $x_j - x' = 0$ or $x_j - x'' = 0$, then $\max\{\| x_k - x(\infty)\|: 1 \leqslant k \leqslant n\} = m(\infty) = 0$, which implies that $x_1 = x_2 = \cdots = x_n$, contrary to hypothesis. Hence, $x_j - x' \neq 0$ and $x_j - x'' \neq 0$. Since X is rotund, $x_j - x' = \alpha(x_j - x'')$ for some $\alpha > 0$. From $\| x_j - x' \| = \| x_j - x'' \| > 0$, we conclude that $\alpha = 1$. Hence, $x_j - x' = x_j - x''$, that is, $x' = x''$, as desired.

THEOREM 3. *Let $x_1, x_2, \ldots, x_n$ be a finite sequence of points in a real, rotund normed linear space X; and let $1 < p < \infty$. Then A_p is continuous and convex in X. If $x_1 = x_2 = \cdots = x_n$ does not hold, then A_p is strictly convex in X. If $x_1 = x_2 = \cdots = x_n$, then A_p is strictly convex on each line not containing x_1 and convex and concave on each closed ray issuing from x_1.*

Proof. Clearly, A_p is continuous. Assume that $x, y \in X$, $x \neq y$, $a > 0$, $b > 0$, and $a + b = 1$. To prove that A_p is convex in X, note that

$$\begin{aligned} A_p(ax + by) &= \left\{(1/n) \sum_{k=1}^{n} \| x_k - (ax + by)\|^p\right\}^{1/p} \\ &= \left\{(1/n) \sum_{k=1}^{n} \| a(x_k - x) + b(x_k - y)\|^p\right\}^{1/p} \\ &\leqslant \left\{(1/n) \sum_{k=1}^{n} (\| a(x_k - x)\| + \| b(x_k - y)\|)^p\right\}^{1/p} \\ &\leqslant \left\{(1/n) \sum_{k=1}^{n} \| a(x_k - x)\|^p\right\}^{1/p} + \left\{(1/n) \sum_{k=1}^{n} \| b(x_k - y)\|^p\right\}^{1/p} \\ &= aA_p(x) + bA_p(y). \end{aligned}$$

Next, let us study strict convexity. Suppose that $A_p(ax + by) = aA_p(x) + bA_p(y)$. Then equality must hold in the last two inequalities. Since X is rotund, we conclude that for each k, there exist nonnegative real numbers, c_k and d_k, such that $c_k + d_k > 0$ and $c_k a(x_k - x) = d_k b(x_k - y)$. Since equality occurs in Minkowski's inequality, there exist nonnegative real numbers c and d such that $c + d > 0$ and $c \| a(x_k - x)\| = d \| b(x_k - y)\|$ for $k = 1, 2,..., n$.

Suppose that $1 \leqslant k \leqslant n$ and $x_k - x = 0$. Then from $c \| a(x_k - x)\| = d \| b(x_k - y)\|$ we conclude that $0 = d \| b(x_k - y)\|$. Hence, $0 = db(x_k - y)$ and $ca(x_k - x) = db(x_k - y)$.

Next, suppose that $1 \leqslant k \leqslant n$ and $x_k - x \neq 0$. Then $d_k \neq 0$. Indeed, if $d_k = 0$, then $c_k a(x_k - x) = d_k b(x_k - y)$ yields $c_k a(x_k - x) = 0$. But $c_k > 0$, since $c_k + d_k > 0$. Thus, $x_k - x = 0$, a contradiction. Likewise, $d \neq 0$. From $c_k a(x_k - x) = d_k b(x_k - y)$ and $c \| a(x_k - x)\| = d \| b(x_k - y)\|$, we conclude that $c/d = \| b(x_k - y)\|/\| a(x_k - x)\| = c_k/d_k$. Thus, $ca(x_k - x) = (dc_k/d_k)\, a(x_k - x) = (d/d_k)\, c_k a(x_k - x) = (d/d_k)\, d_k b(x_k - y) = db(x_k - y)$. Hence, $ca(x_k - x) = db(x_k - y)$ for $k = 1, 2,..., n$.

Next, let us prove that $ca - db \neq 0$. Suppose that $ca = db$ and recall that $a > 0$ and $b > 0$. If $c = 0$, then $d > 0$ and $ca = db$ yields $0 = db > 0$. Hence, $c > 0$ and $ca = db > 0$. Thus, $x_k - x = x_k - y$, that is, $x = y$, a contradiction.

Since $ca - db \neq 0$, $x_k = (cax - dby)/(ca - db)$ for $k = 1, 2,..., n$. Consequently, if $x_1 = x_2 = \cdots = x_n$ does not hold, then $A_p(ax + by) < aA_p(x) + bA_p(y)$, that is, A_p is strictly convex.

Suppose that $x_1 = x_2 = \cdots = x_n$. If $A_p(ax + by) = aA_p(x) + bA_p(y)$, then, as noted above, $x_1 = (cax - dby)/(ca - db)$. This implies that x and y are on a closed ray issuing from x_1, since $x = x_1 + \{db/(ca)\}(y - x_1)$, where $db/(ca) \geqslant 0$ if $c \neq 0$ and $y = x_1 + \{ca/(db)\}(x - x_1)$, where $ca/(db) \geqslant 0$, if $d \neq 0$. Consequently, if x and y are on a line not containing x_1, then $A_p(ax + by) < aA_p(x) + bA_p(y)$.

If $x_1 = x_2 = \cdots = x_n$ and x and y are on a closed ray issuing from x_1, then one can easily verify that $A_p(ax + by) = \| x_1 - (ax + by)\| = \| a(x_1 - x) + b(x_1 - y)\| = a \| x_1 - x \| + b \| x_1 - y \| = aA_p(x) + bA_p(y)$, as desired.

As the reader can see, implicit in the reasoning above is a necessary and sufficient condition for $A_p(ax + by) = aA_p(x) + bA_p(y)$ to hold.

Note that the uniqueness portion of Theorem 1 follows at once from Theorem 3.

COROLLARY. *Let $x_1, x_2,..., x_n$ be a finite sequence of points in a real, rotund normed linear space X; and let $1 < p < \infty$. Then S_p is continuous and is strictly convex in X.*

Proof. First, assume that $x_1 = x_2 = \cdots = x_n$ does not hold. Then, by Theorem 3, A_p is strictly convex in X. Moreover, $S_p(x) = n\{A_p(x)\}^p$ for each $x \in X$. Since one can prove that a strictly increasing convex function of a strictly convex function is strictly convex, it follows that S_p is strictly convex in X.

Next, assume that $x_1 = x_2 = \cdots = x_n$. Then $S_p(x) = n \| x_1 - x \|^p$ for each $x \in X$. Suppose that $x, y \in X$, $x \neq y$, $a > 0$, $b > 0$, and $a + b = 1$. Then $\| x_1 - (ax + by) \| \leqslant a \| x_1 - x \| + b \| x_1 - y \|$, and, as one can prove easily by the previous arguments, if equality holds, x and y are on a closed ray issuing from x_1. Moreover,

$$\{a \| x_1 - x \| + b \| x_1 - y \|\}^p \leqslant a \| x_1 - x \|^p + b \| x_1 - y \|^p,$$

and if equality holds, $\| x_1 - x \| = \| x_1 - y \|$. The last assertion follows from a familiar property of power means [13, p. 26], or from the strict convexity of the function t^p on $[0, \infty)$. Hence,

$$\begin{aligned} S_p(ax + by) &= n \| x_1 - (ax + by) \|^p \\ &\leqslant n\{a \| x_1 - x \| + b \| x_1 - y \|\}^p \\ &\leqslant n\{a \| x_1 - x \|^p + b \| x_1 - y \|^p\} \\ &= aS_p(x) + bS_p(y), \end{aligned}$$

which proves that S_p is convex in X. If $S_p(ax + by) = aS_p(x) + bS_p(y)$, then x and y are on a closed ray issuing from x_1 and are equidistant from x_1, which implies that $x = y$. Since $x \neq y$, S_p is strictly convex in X.

Next, we observe that A_∞ need not be strictly convex in X even if $x_1 = x_2 = \cdots = x_n$ does not hold. Indeed, if $X = R^2$, $n = 2$, $x_1 = (0, 0)$, and $x_2 = (1, 0)$, then A_∞ is convex and concave when restricted to the closed ray issuing from the point $(\frac{1}{2}, \frac{1}{2})$ and passing through the point $(1, 1)$. However, A_∞ is strictly convex on each line that does not pass through x_1 or x_2. More generally, we have:

THEOREM 4. *Let $x_1, x_2, \ldots, x_n$ be a finite sequence of points in a real, rotund normed linear space X. Then A_∞ is continuous and convex in X. If $x_1 = x_2 = \cdots = x_n$ does not hold, then A_∞ is strictly convex on each line containing no x_k. If $x_1 = x_2 = \cdots = x_n$, then $A_\infty = A_p$ for each p, $1 < p < \infty$, and Theorem 3 applies.*

Proof. Clearly, A_∞ is continuous, since the maximum of a finite sequence of continuous functions is continuous.

Suppose that $x, y \in X$, $x \neq y$, $a > 0$, $b > 0$, and $a + b = 1$. Then, for some j,

$$\begin{aligned} A_\infty(ax + by) &= \| x_j - (ax + by) \| \\ &= \| a(x_j - x) + b(x_j - y) \| \\ &\leqslant a \| x_j - x \| + b \| x_j - y \| \\ &\leqslant aA_\infty(x) + bA_\infty(y). \end{aligned}$$

Hence, A_∞ is convex in X.

Assume that $x_1 = x_2 = \cdots = x_n$ does not hold. If $A_\infty(ax + by) = aA_\infty(x) + bA_\infty(y)$, then equality holds in the last two inequalities. Hence, (i) $\| x_j - x \| = A_\infty(x)$, (ii) $\| x_j - y \| = A_\infty(y)$, and by a familiar argument, (iii) x and y are points on a closed ray issuing from x_j. If the line through x and y contains no x_k, then (iii) fails; hence, $A_\infty(ax + by) < aA_\infty(x) + bA_\infty(y)$. This proves that A_∞ is strictly convex on each line containing no x_k.

Conversely, as one can verify, if for some j, (i), (ii), and (iii) hold, then $A_\infty(ax + by) = aA_\infty(x) + bA_\infty(y)$.

Since the pointwise limit of a sequence of convex functions is convex, the convexity of A_∞ follows from that of $A_p(1 < p < \infty)$ and the following result. Note that strict convexity need not be preserved by uniform convergence, as illustrated by the behavior of A_p.

THEOREM 5. *Let X be a real Hilbert space or a finite-dimensional, real, rotund normed linear space; and let $x_1, x_2, \ldots, x_n$ be a finite sequence of points in X. Then S_1 is continuous and convex in X. If $x, y \in X$, $x \neq y$, $a > 0$, $b > 0$, and $a + b = 1$, then $S_1(ax + by) = aS_1(x) + bS_1(y)$ if and only if each x_k lies on the line through x and y but not on the open line segment joining x and y. If the x_k's are not collinear, S_1 is strictly convex in X and attains its infimum, $l(1)$, at a unique point, $x(1)$. If the x_k's are collinear and are relabeled with the subscripts $1, 2,\ldots, n$ so that their linear order corresponds to the order of their subscripts, then $\theta = \{x \in X: S_1(x) = l(1)\}$ is the closed line segment joining $x_{n/2}$ to $x_{(n/2)+1}$ if n is even and is $\{x_{(n+1)/2}\}$ if n is odd. Thus, θ consists of a single point when n is odd and also when n is even and $x_{n/2} = x_{(n/2)+1}$.*

Proof. Suppose that $x, y \in X$, $x \neq y$, $a > 0$, $b > 0$, and $a + b = 1$. Then

$$\begin{aligned} S_1(ax + by) &= \sum_{k=1}^{n} \| a(x_k - x) + b(x_k - y)\| \\ &\leqslant \sum_{k=1}^{n} \{\| a(x_k - x)\| + \| b(x_k - y)\|\} \\ &= aS_1(x) + bS_1(y). \end{aligned}$$

This proves that S_1 is convex in X. If equality holds, then for each k, $\| a(x_k - x) + b(x_k - y)\| = \| a(x_k - x)\| + \| b(x_k - y)\|$. As we have observed before, the last equality is valid if and only if x and y are on a closed ray issuing from x_k. Hence, $S_1(ax + by) = aS_1(x) + bS_1(y)$ if and only if each x_k lies on the line through x and y but not on the open line segment joining x and y. In particular, S_1 is strictly convex if the x_k's are not collinear. Strict convexity, in turn, implies that $\theta = \{x \in X: S_1(x) = l(1)\}$ contains precisely one point. (In virtue of Theorem 1, θ is nonempty.)

Next, suppose all the x_k's lie on some line L. If θ contains at least two points, then, by the first part of the theorem, it follows readily that $\theta \subseteq L$; but conceivably θ may consist of a single point off L. We now prove that always $\theta \subseteq L$. If X is a Hilbert space, then this certainly is the case, since, according to Theorem 1, $\theta \subseteq H \subseteq L$, where H is the convex hull of $\{x_1, x_2, \ldots, x_n\}$.

Suppose that X is finite-dimensional and that u and v are distinct points of L. Furthermore, suppose that $\theta \cap L = \varnothing$. Then $\theta = \{x(1)\}$, $x(1) \notin L$. Let L_1 be the line through the origin, 0, and $v - u$, and consider the two-dimensional subspace, X_1, of X containing $x(1) - u$ and $v - u$. Each $x_k - u$ belongs to L_1, but $x(1) - u$ does not. Hence, $x(1) - u$ does not belong to the convex hull of $\{x_1 - u, x_2 - u, \ldots, x_n - u\}$. Since $\sum_{k=1}^{n} \| x - (x_k - u)\|$ attains its infimum in X_1 at $x(1) - u$, this contradicts Theorem 1. Thus, $\theta \subseteq L$.

We omit the somewhat tedious proof of the last sentence of Theorem 5, since it is patterned after the proof of the minimum property of a median of a finite sequence of real numbers (cf. [2, pp. 32–34]).

THEOREM 6. *Let $x_1, x_2, \ldots, x_n$ be a finite sequence of points in a real normed linear space X. Then, for each $x \in X$, $A_p(x)$ is increasing for $1 \leqslant p \leqslant \infty$ and $\lim_{p\to\infty} A_p(x) = A_\infty(x)$. Moreover, the convergence is uniform on each compact subset of X.*

Proof. The second sentence of Theorem 6 follows immediately from well-known properties of means (cf. [13, pp. 15; 26; 3, pp. 16–17]). The third follows from Dini's theorem [14, p. 205].

THEOREM 7. *Let X be a real Hilbert space or a finite-dimensional, real, rotund normed linear space; and let $x_1, x_2, \ldots, x_n$ be a finite sequence of points in X. Then $m(p)$ is continuous and increasing on $[1, \infty]$. In particular, $m(p) \to m(\infty)$ as $p \to \infty$.*

Preliminary Remark. Theorems 6 and 7 imply that

$$\max_{p\in[1,\infty]} \min_{x\in X} A_p(x) = \min_{x\in X} \max_{p\in[1,\infty]} A_p(x).$$

Proof of Theorem 7. Suppose that $1 \leqslant p_1 < p_2 \leqslant \infty$. Then $m(p_1) = \inf\{A_{p_1}(x): x \in X\} \leqslant A_{p_1}(x(p_2)) \leqslant A_{p_2}(x(p_2)) = m(p_2)$ by Theorems 1 and 6.

Concerning continuity, let $I = [1, b]$ where $1 < b < \infty$. From the proofs of Theorems 1 and 2, we know that there exists a compact set C (take C to be the convex hull of $\{x_1, x_2, \ldots, x_n\}$ if X is a Hilbert space and to be $\{x \in X: \| x \| \leqslant 2A_\infty(0)\}$ otherwise) such that $m(p) = \min\{A_p(x): x \in C\} =$

$A_p(x(p))$ for some $x(p) \in C$ whenever $1 \leqslant p \leqslant \infty$. (We do not claim that $x(1)$ is unique.)

Since $A_p(x)$ is continuous on the compact metric space $I \times C$, it is uniformly continuous there. Hence, given $\epsilon > 0$, there exists a $\delta > 0$ such that $|A_{p'}(x) - A_{p''}(x)| < \epsilon$ if $p', p'' \in I$, $|p' - p''| < \delta$, and $x \in C$. For such p' and p'',

$$-\epsilon < A_{p'}(x(p')) - A_{p''}(x(p')) \leqslant A_{p'}(x(p')) - A_{p''}(x(p''))$$
$$\leqslant A_{p'}(x(p'')) - A_{p''}(x(p'')) < \epsilon.$$

Hence, $|m(p') - m(p'')| = |A_{p'}(x(p')) - A_{p''}(x(p''))| < \epsilon$. Therefore, $m(p)$ is uniformly continuous in I. (For the convenience of the reader, we have repeated something here that is essentially well known (see [21, pp. 101, 295].)

Finally, let us prove that $m(p) \to m(\infty)$ as $p \to \infty$. According to Theorem 6, A_p converges uniformly on C to A_∞ as $p \to \infty$. Suppose that $\epsilon > 0$. Then there exists a $p_\epsilon \in (1, \infty)$ such that $0 \leqslant A_\infty(x) - A_p(x) < \epsilon$ if $x \in C$ and $p_\epsilon < p < \infty$. Hence, $m(\infty) = A_\infty(x(\infty)) \leqslant A_\infty(x(p)) < A_p(x(p)) + \epsilon = m(p) + \epsilon$ if $p_\epsilon < p < \infty$. Thus, $0 \leqslant m(\infty) - m(p) < \epsilon$ if $p_\epsilon < p < \infty$; consequently, $m(p) \to m(\infty)$ as $p \to \infty$.

THEOREM 8. *Let X be a real Hilbert space or a finite-dimensional, real, rotund normed linear space; and let $x_1, x_2, \ldots, x_n$ be a finite sequence of points in X. Then $x(p)$ is continuous on $(1, \infty]$ and in particular, $x(p) \to x(\infty)$ as $p \to \infty$. Moreover, $x(p)$ converges to a limit, $x(1)$, as $p \to 1+$ and $A_1(x(1)) = m(1)$.*

Proof. Suppose that $x(p)$ is not continuous at some point $p \in (1, \infty)$. Let C be the compact set introduced in the proof of Theorem 7. Then there exists a point $x' \in C$ and a sequence of points $p_1, p_2, p_3, \ldots$ in $(1, \infty)$ such that $x' \neq x(p)$, $p_k \to p$, and $x(p_k) \to x'$ as $k \to \infty$. Now, $m(p_k) \to m(p) = A_p(x(p))$ as $k \to \infty$, by Theorem 7. Moreover, $m(p_k) = A_{p_k}(x(p_k)) \to A_p(x')$ by the continuity of $A_p(x)$ on $[1, \infty) \times X$. Thus, $A_p(x') = A_p(x(p))$. According to Theorem 1, $x(p) = x'$, a contradiction.

Next, consider the case $p = 1$. If the x_k's are not collinear, then, according to Theorem 5, A_1 attains its infimum at a unique point of X, that is, $\theta = \{x \in X: A_1(x) = m(1)\}$ contains exactly one point, $x(1)$. In this case, one proves that $x(p) \to x(1)$ as $p \to 1+$ by the same argument that was used above.

Now, assume the x_k's all lie on some line L and let u, v be distinct points of L. According to Theorem 5, $\theta \subseteq L$. If $1 < p < \infty$, then the proof of Theorem 5, with θ replaced by $\{x(p)\}$, shows that $x(p) \in L$. For $k = 1, 2, \ldots, n$, let r_k be the real number satisfying $x_k = u + r_k(v - u)$ and for each $p > 1$, let $r(p)$ be the real number satisfying $x(p) = u + r(p)(v - u)$. Let $p > 1$. If r is an arbitrary real number, then

$$\| v - u \|^p \sum_{k=1}^{n} | r_k - r(p)|^p = \sum_{k=1}^{n} \| x_k - x(p)\|^p$$

$$\leqslant \sum_{k=1}^{n} \| x_k - \{u + r(v - u)\}\|^p$$

$$= \| v - u \|^p \sum_{k=1}^{n} | r_k - r |^p,$$

so that $r(p)$ minimizes $\sum_{k=1}^{n} | r_k - r |^p$.

By Jackson's theorem [15], $r(p)$ converges to some (finite) number, $r(1)$, as $p \to 1+$ and $r(1)$ is a median of $r_1, r_2, \ldots, r_n$. Hence, $\lim_{p\to 1+} x(p) = u + r(1)(v - u)$; we denote this limit $x(1)$.

To prove that $A_1(x(1)) = m(1)$, first recall that $\sum_{k=1}^{n} | r_k - r(1)| \leqslant \sum_{k=1}^{n} | r_k - r |$ for each real number r, since $r(1)$ is a median of $r_1, r_2, \ldots, r_n$. This implies that $\sum_{k=1}^{n} \| x_k - x(1)\| \leqslant \sum_{k=1}^{n} \| x_k - x \|$ for each $x \in L$. Since $\theta \cap L \neq \varnothing$, there exists a point $x' \in L$ such that $\sum_{k=1}^{n} \| x_k - x' \| \leqslant \sum_{k=1}^{n} \| x_k - x \|$ for each $x \in X$. Hence, $\sum_{k=1}^{n} \| x_k - x(1)\| \leqslant \sum_{k=1}^{n} \| x_k - x \|$ for each $x \in X$, as desired. This conclusion also follows, upon letting $p \to 1+$, from $\sum_{k=1}^{n} \| x_k - x(p)\|^p \leqslant \sum_{k=1}^{n} \| x_k - x \|^p$, holding for each $x \in X$.

Finally, let us prove that $x(p) \to x(\infty)$ as $p \to \infty$. Suppose not. Then, since $\{x(p)\colon 1 < p \leqslant \infty\}$ is contained in a compact subset of X, there exist a sequence, $p_1, p_2, p_3, \ldots$, of real numbers and a point $x' \in X$ such that $x' \neq x(\infty)$, $p_k \to \infty$, and $x(p_k) \to x'$ as $k \to \infty$. According to Theorem 7, $m(p_k) \to m(\infty) = A_\infty(x(\infty))$. Moreover, $m(p_k) = A_{p_k}(x(p_k)) \to A_\infty(x')$. To prove this last assertion, we first note that $A_\infty(x') - A_\infty(x(p_k)) \to 0$ as $k \to \infty$, since $x(p_k) \to x'$ and A_∞ is continuous. Next, we note that $A_\infty(x(p_k)) - A_{p_k}(x(p_k)) \to 0$, since $x(p_k) \in C$ for each k, and $A_p(x) \to A_\infty(x)$, uniformly on C, as $p \to \infty$. (C is the compact set defined in the proof of Theorem 7.) Consequently,

$$A_\infty(x') - A_{p_k}(x(p_k))$$
$$= \{A_\infty(x') - A_\infty(x(p_k))\} + \{A_\infty(x(p_k)) - A_{p_k}(x(p_k))\} \to 0$$

as $k \to \infty$. Hence, $A_\infty(x(\infty)) = A_\infty(x')$. According to Theorem 2, $x(\infty) = x'$, a contradiction.

LEMMA 2. *Let X be a real Hilbert space or a finite-dimensional, real, rotund normed linear space; and let $x_1, x_2, \ldots, x_n$ be a finite sequence of points in X.*

Then, if $1 < p_1 < p_2 < \infty$, *we have*

$$m(\infty) \leqslant \{l(p_2)\}^{1/p_2} \leqslant \{l(p_1)\}^{1/p_1} \leqslant n^{1/p_1}\, m(\infty).$$

Proof. Since $m(p) \leqslant m(\infty)$ for each $p \in [1, \infty]$, according to Theorem 7, and since $l(p) = n\{A_p(x(p))\}^p = n\{m(p)\}^p$, we infer that $l(p_1) \leqslant n\{m(\infty)\}^{p_1}$ if $1 < p_1 < \infty$. Hence, $\{l(p_1)\}^{1/p_1} \leqslant n^{1/p_1} m(\infty)$ if $1 < p_1 < \infty$.

Suppose that $1 < p_1 < p_2 < \infty$. Then $l(p_2) \leqslant \sum_{k=1}^{n} \| x_k - x(p_1)\|^{p_2}$. Hence,

$$\begin{aligned} \{l(p_2)\}^{1/p_2} &\leqslant \left\{\sum_{k=1}^{n} \| x_k - x(p_1)\|^{p_2}\right\}^{1/p_2} \\ &\leqslant \left\{\sum_{k=1}^{n} \| x_k - x(p_1)\|^{p_1}\right\}^{1/p_1} \\ &= \{l(p_1)\}^{1/p_1} \end{aligned}$$

(cf. [13, p. 28; 3, p. 18].) According to a familiar fact from the theory of inequalities, $\max\{\| x_k - x \|: 1 \leqslant k \leqslant n\} \leqslant \{\sum_{k=1}^{n} \| x_k - x \|^{p_2}\}^{1/p_2}$ for each $x \in X$ (cf. [13, pp. 28–29; 3, p. 18].) Hence,

$$\begin{aligned} m(\infty) &\leqslant \max\{\| x_k - x(p_2)\| : 1 \leqslant k \leqslant n\} \\ &\leqslant \left\{\sum_{k=1}^{n} \| x_k - x(p_2)\|^{p_2}\right\}^{1/p_2} \\ &= \{l(p_2)\}^{1/p_2}, \end{aligned}$$

as desired.

THEOREM 9. *Let X be a real Hilbert space or a finite-dimensional, real, rotund normed linear space; and let* $x_1, x_2, \ldots, x_n$ *be a finite sequence of points (at least two of which are distinct) in X. If* $m(\infty) < 1$, *then* $\lim_{p\to\infty} l(p) = 0$; *if* $m(\infty) > 1$, *then* $\lim_{p\to\infty} l(p) = \infty$; *and if* $m(\infty) = 1$, *then* $1 \leqslant l(p) \leqslant n$ *for each* $p \in (1, \infty)$. *If* $l(p') < 1$ *for some* $p' \in (1, \infty)$, *then* $l(p)$ *is strictly decreasing on* $[p', \infty)$; *and if* $l(p'') > n$ *for some* $p'' \in (1, \infty)$, *then* $l(p)$ *is strictly increasing on* $[p'', \infty)$. *In particular, if* $m(\infty) \neq 1$, *then* $l(p)$ *is strictly monotonic for all sufficiently large values of* p.

Proof. From the first and last inequalities in the last line of Lemma 2, we conclude that $\{m(\infty)\}^p \leqslant l(p) \leqslant n\{m(\infty)\}^p$ if $p \in (1, \infty)$. Consequently, if $m(\infty) < 1$, then $l(p) \to 0$ as $p \to \infty$; if $m(\infty) > 1$, then $l(p) \to \infty$ as $p \to \infty$; and if $m(\infty) = 1$, then $1 \leqslant l(p) \leqslant n$ for each $p \in (1, \infty)$.

Next, assume that $l(p') < 1$ for some $p' \in (1, \infty)$. Then, according to Lemma 2, $l(p) \leqslant \{l(p')\}^{p/p'}$ if $p' \leqslant p < \infty$. Consequently, if $p' \leqslant p_1 < p_2 < \infty$, then $l(p_2) \leqslant \{l(p_1)\}^{p_2/p_1} = l(p_1)\{l(p_1)\}^{(p_2-p_1)/p_1} < l(p_1)$.

Finally, assume that $n < l(p'')$ for some $p'' \in (1, \infty)$. Then

$$1 < \{(1/n)\, l(p'')\}^{1/p''} = m(p'').$$

If $p'' \leqslant p_1 < p_2 < \infty$, then $1 < m(p'') \leqslant m(p_1) \leqslant m(p_2)$, according to Theorem 7. Hence, $l(p_1) = n\{m(p_1)\}^{p_1} \leqslant n\{m(p_2)\}^{p_1} < n\{m(p_2)\}^{p_2} = l(p_2)$.

COROLLARY. *Assume the first sentence of Theorem* 9. *As usual, let* $m(2)$ *denote the "standard deviation"* $\{(1/n)\sum_{k=1}^{n} \| x_k - x(2)\|^2\}^{1/2}$. *If* $m(2) < n^{-1/2}$, *then* $l(p)$ *is strictly decreasing on* $[2, \infty)$; *and if* $1 < m(2)$, *then* $l(p)$ *is strictly increasing on* $[2, \infty)$. *Moreover, if* $m(\infty) < 1$, *then* $l(p)$ *is strictly decreasing on* $[(\log n)/\log(1/m(\infty)), \infty)$ *and if* $m(\infty) > 1$, *then* $l(p)$ *is strictly increasing on* $[(\log n)/\log m(\infty), \infty)$.

Proof. The first two assertions follow from Theorem 9 and the fact that $m(2) = \{(1/n)\, l(2)\}^{1/2}$.

Suppose that $m(\infty) < 1$. From Lemma 2 we know that $l(p) \leqslant n\{m(\infty)\}^p$ if $p \in (1, \infty)$. Thus, if $n\{m(\infty)\}^p < 1$, then $l(p) < 1$. But $n\{m(\infty)\}^p < 1$ if and only if $(\log n)/\log(1/m(\infty)) < p$. The monotonicity of $l(p)$ follows from Theorem 9.

Suppose that $m(\infty) > 1$. From Lemma 2 we know that $\{m(\infty)\}^p \leqslant l(p)$ if $p \in (1, \infty)$. Thus, if $n < \{m(\infty)\}^p$, then $n < l(p)$. But $n < \{m(\infty)\}^p$ if and only if $(\log n)/\log m(\infty) < p$. The monotonicity of $l(p)$ follows from Theorem 9.

THEOREM 10. *Let X be a real Hilbert space or a finite-dimensional, real, rotund normed linear space; and let* $x_1, x_2, ..., x_n$ *be a finite sequence of points in X. Then* $\{l(p)\}^{1/p}$ *is monotonically decreasing on* $(1, \infty)$ *to the limit* $m(\infty)$. *Moreover,* $0 \leqslant \{l(p)\}^{1/p} - m(\infty) \leqslant (n^{1/p} - 1)\, m(\infty) \leqslant ((n-1)/p)\, m(\infty)$ *for each* $p \in (1, \infty)$.

Proof. All of the assertions except the last inequality follow immediately from Lemma 2. The fact that $n^{1/p} - 1 \leqslant (n-1)/p$ whenever $p \in (1, \infty)$ follows from [13, p. 40].

Next, we estimate how fast $m(p) \to m(\infty)$ as $p \to \infty$. It turns out that $p\{m(\infty) - m(p)\}$ remains bounded as $p \to \infty$. The following result sharpens a portion of Theorem 7 by adding quantitative information. It also gives complementary inequalities (cf. [8]).

THEOREM 11. *Let X be a real Hilbert space or a finite-dimensional, real, rotund normed linear space; and let* $x_1, x_2, ..., x_n$ *be a finite sequence of points (at least two of which are distinct) in X. Then*

$$n^{-1/p_1}/n^{-1/p_2} \leqslant m(p_1)/m(p_2) \leqslant 1$$

if $1 < p_1 < p_2 < \infty$. *Moreover,*

$$0 \leqslant m(\infty) - m(p) \leqslant m(\infty)\{1 - n^{-1/p}\} \leqslant ((\log n)/p)\, m(\infty)$$

if $p \in (1, \infty)$.

Proof. Using the fact that $\{l(p)\}^{1/p} = n^{1/p} m(p)$ if $p \in (0, \infty)$, we conclude from Lemma 2 that $m(\infty) \leqslant n^{1/p_2} m(p_2) \leqslant n^{1/p_1} m(p_1) \leqslant n^{1/p_1} m(\infty)$ if $1 < p_1 < p_2 < \infty$. From $n^{1/p_2} m(p_2) \leqslant n^{1/p_1} m(p_1)$ we infer that

$$n^{-1/p_1}/n^{-1/p_2} \leqslant m(p_1)/m(p_2).$$

The fact that $m(p_1)/m(p_2) \leqslant 1$ follows from Theorem 7. From $m(\infty) \leqslant n^{1/p_1} m(p_1) \leqslant n^{1/p_1} m(\infty)$ we see that

$$0 \leqslant m(\infty) - m(p_1) \leqslant m(\infty) - n^{-1/p_1} m(\infty) = m(\infty)\{1 - n^{-1/p_1}\}$$

if $p_1 \in (1, \infty)$.

To prove that $1 - n^{-1/p} \leqslant (\log n)/p$, apply the mean-value theorem to the function n^{-x} on the interval $[0, 1/p]$.

3. Some Specialized Results

In this section we restrict our attention to the case $X = R$. We give new proofs of some previous results, and we prove some new ones.

Suppose that $1 < p < \infty$, and let $f(x) = |x|^p$ for each real number x. Then

$$f'(x) = \begin{cases} p\,|x|^{p-1} & \text{if } x \geqslant 0 \\ -p\,|x|^{p-1} & \text{if } x < 0 \end{cases} = px\,|x|^{p-2}$$

(meaning 0 when $x = 0$). Clearly, f' is strictly increasing on R.

Assume throughout this section that $x_1 \leqslant x_2 \leqslant \cdots \leqslant x_n$ and that $x_1 \neq x_n$. We are interested in $S_p(x) = \sum_{k=1}^{n} |x_k - x|^p = \sum_{k=1}^{n} f(x - x_k)$. Since $S_p'(x) = \sum_{k=1}^{n} f'(x - x_k) = \sum_{k=1}^{n} p(x - x_k)\,|x_k - x|^{p-2}$, it is obvious that S_p' is strictly increasing on R, $S_p'(x) < 0$ if $x \leqslant x_1$, and $S_p'(x) > 0$ if $x \geqslant x_n$.

This proves that, for each $p \in (1, \infty)$, S_p *is strictly convex in* R, *that* S_p *attains its infimum over* R *at a unique point* $x(p)$, *and that* $x(p)$ *is in the convex hull of* $\{x_1, x_2, \ldots, x_n\}$.

Next, let us prove that $\lim_{p\to\infty} x(p) = x(\infty) = (x_1 + x_n)/2 = a$. Let r be the smallest k with $x_k > x_1$. Let $0 < \epsilon < (x_n - x_1)/2$. For $k = r, r + 1, \ldots, n$, let $m_k = \max\{|x - x_k|/|x - x_1| : a + \epsilon \leqslant x \leqslant x_n\}$.

Note that each m_k is <1. Now, for each $p \in (2, \infty)$ and for each real number $x \neq x_1$,

$$\begin{aligned}\frac{1}{p} S_p'(x) &= \sum_{k=1}^{n} (x - x_k) \mid x_k - x \mid^{p-2} \\ &= (x - x_1) \mid x_1 - x \mid^{p-2} \left\{ r - 1 + \sum_{k=r}^{n} \frac{x - x_k}{x - x_1} \left| \frac{x_k - x}{x_1 - x} \right|^{p-2} \right\}.\end{aligned}$$

Hence, for each $x \in [a + \epsilon, x_n]$,

$$\begin{aligned}\left| \frac{1}{p} S_p'(x) \right| &\geqslant \mid x - x_1 \mid \mid x_1 - x \mid^{p-2} \left\{ r - 1 - \left| \sum_{k=r}^{n} \frac{x - x_k}{x - x_1} \left| \frac{x_k - x}{x_1 - x} \right|^{p-2} \right| \right\} \\ &\geqslant \mid x_1 - x \mid^{p-1} \left\{ r - 1 - \sum_{k=r}^{n} \left| \frac{x_k - x}{x_1 - x} \right|^{p-1} \right\} \\ &\geqslant \mid x_1 - x \mid^{p-1} \left\{ r - 1 - \sum_{k=r}^{n} m_k^{p-1} \right\} \\ &> 0\end{aligned}$$

for all (finite) $p \geqslant$ some (finite) p_0, independent of x. Hence, $x(p) < a + \epsilon$ if $p \geqslant p_0$. Similarly, there exists a (finite) p_0' such that $x(p) > a - \epsilon$ if $\infty > p \geqslant p_0'$. Hence, $x(p) \to a$ as $p \to \infty$.

Next, let us prove that $\lim_{p\to\infty} m(p) = m(\infty) = (x_n - x_1)/2$. As above, let $a = (x_1 + x_n)/2$. Also, let $\epsilon > 0$. Since $\lim_{p\to\infty} x(p) = a$, there exists a real number, $N > 1$, such that $a - \epsilon < x(p) < a + \epsilon$ if $p \geqslant N$. Clearly, $\mid x(p) - x_k \mid < [(x_n - x_1)/2] + \epsilon$ if $1 \leqslant k \leqslant n$ and $p \geqslant N$. Hence, $A_p(x(p)) < [(x_n - x_1)/2] + \epsilon$ if $p \geqslant N$.

On the other hand, $\mid x_1 - x(p) \mid \geqslant (x_n - x_1)/2$ or $\mid x_n - x(p) \mid \geqslant (x_n - x_1)/2$ must hold for each $p > 1$. Thus,

$$\begin{aligned}m(p) &= \left\{ \frac{1}{n} \sum_{k=1}^{n} \mid x_k - x(p) \mid^p \right\}^{1/p} \\ &\geqslant \left(\frac{1}{n} \right)^{1/p} \cdot \frac{x_n - x_1}{2}\end{aligned}$$

for each $p \in (1, \infty)$. Thus,

$$\left(\frac{1}{n} \right)^{1/p} \cdot \frac{x_n - x_1}{2} \leqslant m(p) \leqslant \frac{x_n - x_1}{2} + \epsilon$$

if $p \geqslant N$. Since $(1/n)^{1/p} \to 1$ as $p \to \infty$, it follows that $[(x_n - x_1)/2] - \epsilon \leqslant m(p) \leqslant [(x_n - x_1)/2] + \epsilon$ if $p \geqslant N_\epsilon \geqslant N$. Thus, $\lim_{p\to\infty} m(p) = m(\infty)$.

If $m(\infty) = 1$, then by Theorem 9, $1 \leqslant l(p) \leqslant n$ for each $p \in (1, \infty)$. We now prove that, if $X = R$ and $m(\infty) = 1$, then $l(p)$ converges as $p \to \infty$; and we determine the limit.

THEOREM 12. *Suppose that* $x_1, x_2, \ldots, x_n$ *are real numbers and* s *and* t *are positive integers such that* $x_1 = x_2 = \cdots = x_s < x_{s+1} \leqslant x_{s+2} \leqslant \cdots \leqslant x_{n-t-1} \leqslant x_{n-t} < x_{n-t+1} = \cdots = x_n = x_1 + 2$. (*Included is the case* $x_1 = x_2 = \cdots = x_s < x_{s+1} = \cdots = x_n = x_1 + 2$, *with* $t = n - s$.) *Then* $\lim_{p\to\infty} l(p) = 2\{st\}^{1/2}$.

Proof. First, let us consider the above simple case. Clearly, we can assume $x_1 = 0$, $x_n = 2$. Then, if $1 < p < \infty$ and $0 \leqslant x \leqslant 2$, $S_p(x) = \sum_{k=1}^{n} | x_k - x |^p = sx^p + t(2 - x)^p$ and $S_p'(x) = psx^{p-1} - pt(2 - x)^{p-1}$. Clearly, $S_p'(x) = 0$ if and only if $x = x(p) = 2/\{1 + (s/t)^{1/(p-1)}\}$. (Note that $x(p) \to 1 = x(\infty)$ as $p \to \infty$, as it should.) Set $\alpha = s/t$. Then

$$\begin{aligned} l(p) &= S_p(x(p)) \\ &= \alpha t\{x(p)\}^p + t\{2 - x(p)\}^p \\ &= t(\alpha + \alpha^{p/(p-1)}) \left\{\frac{2}{1 + \alpha^{1/(p-1)}}\right\}\left\{\frac{2}{1 + \alpha^{1/(p-1)}}\right\}^{p-1}. \end{aligned}$$

Next, we note that

$$\lim_{p\to\infty} \left\{\frac{2}{1 + \alpha^{1/(p-1)}}\right\}^{p-1} = \alpha^{-1/2},$$

since

$$\left\{\frac{2}{1 + \alpha^{1/(p-1)}}\right\}^{p-1} = \exp\{(p - 1) \log[2\{1 + \alpha^{1/(p-1)}\}^{-1}]\}$$

and

$$\begin{aligned} \lim_{p\to\infty} \frac{\log[2\{1 + \alpha^{1/(p-1)}\}^{-1}]}{(1/(p - 1))} &= \lim_{q\to 0+} q^{-1} \log \frac{2}{1 + \alpha^q} \\ &= \left[\frac{d}{dq} \log \frac{2}{1 + \alpha^q}\right]_{q=0} \\ &= -\tfrac{1}{2} \log \alpha \\ &= \log(\alpha^{-1/2}). \end{aligned}$$

Thus, $\lim_{p\to\infty} l(p) = t(2\alpha)\, \alpha^{-1/2} = 2\{st\}^{1/2}$.

Next, suppose $s < n - t$. For each $p \in (1, \infty)$, let $\hat{x}(p)$ be the value of x for which $\hat{S}_p(x) = \sum_{k=1}^{s} | x_k - x |^p + \sum_{k=n-t+1}^{n} | x_k - x |^p$ is minimal, and let $\hat{l}(p) = \hat{S}_p(\hat{x}(p))$.

Then, for each $p \in (1, \infty)$,

$$\begin{aligned}\hat{l}(p) &\leqslant \hat{S}_p(x(p)) \\ &\leqslant \sum_{k=1}^{n} | x_k - x(p)|^p \\ &= l(p) \\ &\leqslant \sum_{k=1}^{n} | x_k - \hat{x}(p)|^p \\ &= \hat{S}_p(\hat{x}(p)) + \sum_{s<k<n-t+1} | x_k - \hat{x}(p)|^p \\ &= \hat{l}(p) + \sum_{s<k<n-t+1} | x_k - x(p)|^p.\end{aligned}$$

Thus, $\hat{l}(p) \leqslant l(p) \leqslant \hat{l}(p) + \sum_{s<k<n-t+1} | x_k - \hat{x}(p)|^p$ for each $p \in (1, \infty)$. The last sum approaches 0 as $p \to \infty$, since $\hat{x}(p) \to (x_1 + x_n)/2$, $| x_k - \hat{x}(p)| \to | x_k - [(x_1 + x_n)/2]| < 1$, and hence $| x_k - \hat{x}(p)|^p \to 0$ if $s < k < n - t + 1$. Moreover, as proved above, $\hat{l}(p) \to 2\{st\}^{1/2}$ as $p \to \infty$. Finally, since $l(p)$ is bounded by two quantities approaching the common limit $2\{st\}^{1/2}$, we conclude that $l(p) \to 2\{st\}^{1/2}$ as $p \to \infty$.

4. Conclusion

Scattered throughout the literature are numerous results that are loosely related to this paper. For example, the Fermat–Steiner problem for a tetrahedron, that is, the case when $X = R^3$, $n = 4$, $p = 1$, and x_1, x_2, x_3, x_4 are not coplaner, has been treated (cf. [9, p. 359]). For other related results, consult [6].

For the sake of completeness, we now prove the following simple result.

THEOREM 13. *Let $x_1, x_2, \ldots, x_n$ be a finite sequence of points in a real inner product space X. Then $S_2(x) = \sum_{k=1}^{n} \| x_k - x \|^2$ is minimal if and only if $x = x(2) = (1/n) \sum_{k=1}^{n} x_k$.*

Proof. Let $m = (1/n) \sum_{k=1}^{n} x_k$, and let the sign $\langle\,,\rangle$ denote the inner product in X. Then

$$\begin{aligned}\| x_k - x \|^2 &= \langle x_k - x, x_k - x\rangle \\ &= \langle (x_k - m) + (m - x), (x_k - m) + (m - x)\rangle \\ &= \langle x_k - m, x_k - m\rangle + 2\langle x_k - m, m - x\rangle + \langle m - x, m - x\rangle \\ &= \| x_k - m \|^2 + 2\langle x_k - m, m - x\rangle + \| m - x \|^2.\end{aligned}$$

Addition yields $\sum_{k=1}^{n} \| x_k - x \|^2 = \sum_{k=1}^{n} \| x_k - m \|^2 + n \| m - x \|^2$, which is a "generalization" of the Steiner transfer theorem [7, p. 439]. The desired conclusion is obvious.

Theorem 13 does not hold for real, rotund normed linear spaces in general. Consider the space l_3^2 (see Section 1). Let $x_1 = (0, 0)$, $x_2 = (1, 0)$, and $x_3 = (0, 2)$. Then $x(2) \neq (\frac{1}{3}) \sum_{k=1}^{n} x_k = (\frac{1}{3}, \frac{2}{3})$. To prove this, it suffices to show that

$$\frac{\partial F}{\partial u}\left(\frac{1}{3}, \frac{2}{3}\right) \neq 0,$$

where

$$F(u, v) = \{| u |^3 + | v |^3\}^{2/3} + \{| u - 1 |^3 + | v |^3\}^{2/3} + \{| u |^3 + | v - 2 |^3\}^{2/3}.$$

Now, if $0 < u < 1$ and $0 < v < 2$, then

$$\frac{\partial F}{\partial u}(u, v) = \frac{2}{3}\{u^3 + v^3\}^{-1/3} 3u^2 + \frac{2}{3}\{(1 - u)^3 + v^3\}^{-1/3} 3(1 - u)^2 (-1) + \frac{2}{3}\{u^3 + (2 - v)^3\}^{-1/3} 3u^2.$$

Hence,

$$\frac{\partial F}{\partial u}\left(\frac{1}{3}, \frac{2}{3}\right) = \frac{2}{3}\left\{\frac{1}{9^{1/3}} - \frac{2}{2^{1/3}} + \frac{1}{65^{1/3}}\right\} < 0.$$

One might want to extend the concepts and results of this paper from the case of a finite sequence $x_1, x_2, \ldots, x_n$ to a continuous setting. To avoid tedium, we confine our attention to only one such result.

THEOREM 14. *Let μ be a nondegenerate, nonnegative real Borel measure on a compact subset $K \neq \varnothing$ of R^m where $m \geqslant 1$ and let H denote the convex hull of K. Then, for each $p \in (1, \infty)$, there exists a unique point $x(p)$, in R^m, such that*

$$\int_K \| u - x(p)\|^p \, d\mu(u) = \inf \left\{\int_K \| u - x \|^p \, d\mu(u) : x \in R^m\right\}$$

and $x(p) \in H$.

Proof. Let $p \in (1, \infty)$ and recall that H is compact (cf. [18, p. 21; 5, p. 140]). If $x \in R^m - H$, let x^* denote the unique point of H that is closest to x. Then [22] $\| u - x^* \| < \| u - x \|$ for each $u \in K$. Hence, $\int_K \| u - x^* \|^p \, d\mu(u) \leqslant \int_K \| u - x \|^p \, d\mu(u)$. This proves that it suffices to minimize

$$A_p(x) = \left\{\frac{1}{\mu(K)} \int_K \| u - x \|^p \, d\mu(u)\right\}^{1/p}$$

as x ranges over H. Since A_p is continuous (to prove continuity, use Lebesgue's dominated convergence theorem) on the compact set H, the infimum, $m(p)$, is attained at a point $x(p) \in H$. To prove that $x(p)$ is unique, suppose that x', $x'' \in R^m$ and that $m(p) = A_p(x') = A_p(x'')$. Then, by Minkowski's inequality,

$$\begin{aligned}
A_p\left(\frac{1}{2}(x' + x'')\right) &= \left\{\frac{1}{\mu(K)} \int_K \left\| \frac{1}{2}(u - x') + \frac{1}{2}(u - x'')\right\|^p d\mu(u)\right\}^{1/p} \\
&\leqslant \frac{1}{2}\left\{\frac{1}{\mu(K)} \int_K (\| u - x' \| + \| u - x'' \|)^p \, d\mu(u)\right\}^{1/p} \\
&\leqslant \frac{1}{2}\left[\left\{\frac{1}{\mu(K)} \int_K \| u - x' \|^p \, d\mu(u)\right\}^{1/p} \right. \\
&\qquad \left. + \left\{\frac{1}{\mu(K)} \int_K \| u - x'' \|^p \, d\mu(u)\right\}^{1/p}\right] \\
&= \frac{1}{2}\{A_p(x') + A_p(x'')\} \\
&= m(p).
\end{aligned}$$

Now, $m(p) \leqslant A_p((1/2)(x' + x''))$ by the definition of $m(p)$; hence, equality signs hold in the last three inequalities. Therefore, there exist nonnegative real functions $c(u)$ and $d(u)$ defined on K such that $c(u) + d(u) > 0$ and $c(u)(u - x') = d(u)(u - x'')$ μ a.e. on K. Since equality occurs in Minkowski's inequality, there exist nonnegative real numbers c and d such that $c + d > 0$ and $c \| u - x' \| = d \| u - x'' \|$ μ a.e. on K. Assume (as we may) that μ is not concentrated on a subset of K containing precisely one point. Then the last equality and $m(p) = A_p(x') = A_p(x'') > 0$ imply that $c = d > 0$ and that $\| u - x' \| = \| u - x'' \|$ μ a.e. on K. Since $c(u)(u - x') = d(u)(u - x'')$ μ a.e. on K and $\mu(K) > 0$, there exists a point $u' \in K$ such that $\| u' - x' \| = \| u' - x'' \|$ and $c(u')(u' - x') = d(u')(u' - x'')$. If $\| u' - x' \| = \| u' - x'' \| = 0$, then $x' = u' = x''$, as desired. If $\| u' - x' \| = \| u' - x'' \| \neq 0$, then $c(u')(u' - x') = d(u')(u' - x'')$ yields $c(u') \| u' - x' \| = d(u') \| u' - x'' \|$, $c(u') = d(u') > 0$, $u' - x' = u' - x''$, and $x' = x''$, as desired.

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